

# DIOPHANTINE EXPONENTS FOR STANDARD LINEAR ACTIONS OF $SL_2$ OVER DISCRETE RINGS IN $\mathbb{C}$

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**ABSTRACT.** We give upper and lower bounds for various Diophantine exponents associated with the standard linear actions of  $SL_2(\mathcal{O}_K)$  on the punctured complex plane  $\mathbb{C}^2 \setminus \{0\}$ , where  $K$  is a number field whose ring of integers  $\mathcal{O}_K$  is discrete and within a unit distance of any complex number. The results are similar to those of Laurent and Nogueira for  $SL_2(\mathbb{Z})$  action on  $\mathbb{R}^2 \setminus \{0\}$  albeit for us, uniformly nice bounds are obtained only outside of a set of null measure.

## 1. INTRODUCTION

The set  $\mathbb{Q}$  of rational numbers is dense in  $\mathbb{R}$ . However, one of the first works which tried to quantify this density came only in the nineteenth century from Dirichlet who stated that for any real number  $\theta$  and all  $Q > 1$ , there exist integers  $p$  and  $q$ ,  $1 \leq q < Q$  such that

$$(1.1) \quad |q\theta - p| < \frac{1}{Q} < \frac{1}{q}.$$

This is a consequence of the pigeon-hole principle (also known as Dirichlet's box principle). The inhomogeneous version was given by Minkowski using geometry of numbers. For any  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  and  $\alpha \notin \mathbb{Z}\theta + \mathbb{Z}$ , there exist integers  $p$  and  $q$  for which

$$(1.2) \quad |q\theta - \alpha - p| < \frac{1}{4|q|}.$$

Other than the fact that this second statement is only true for irrational  $\theta$ , the error estimate is also weak here than in Dirichlet's theorem where it is in terms of  $Q^{-1} < q^{-1}$  ( $q < Q$ ). If we now take two such inequalities with different  $\alpha$ 's, we are in the realm of simultaneous inhomogeneous Diophantine approximation [2]. Otherwise said, we are looking for infinitely many integral solutions  $(p_1, q_1, p_2, q_2)$  to the system of inequalities

$$(1.3) \quad \max\{|q_1\theta - p_1 - \alpha_1|, |q_2\theta - p_2 - \alpha_2|\} < \varepsilon.$$

An extra demand that the pairs  $(q_1, p_1)$  and  $(q_2, p_2)$  be primitive can be fulfilled by asking that the matrix

$$(1.4) \quad \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In recent times, mathematicians have been interested in understanding more generally the nature of dense orbits for the action of a group  $G$  on a homogeneous space  $X$ . In this respect, see the works of Ghosh, Gorodnik, and Nevo [5, 6] where they relate the rate of

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*Date:* September 7, 2015.

*2010 Mathematics Subject Classification.* Primary 11J20, 11J13, 11A55; Secondary 22Fxx.

*Key words and phrases.* Diophantine approximation, asymptotic and uniform exponents, lattice subgroups.

The author thanks his supervisor Dr. Anish Ghosh for suggesting the problem, reading the various preliminary versions of the manuscript and for being a constant source of encouragement. Financial support from CSIR, Govt. of India under SPM-07/858(0199)/2014-EMR-I is duly acknowledged.

approximation by ‘rational points’ on a homogeneous space  $X$  of a semisimple group  $G$  to the automorphic representations of  $G$  and compute the exact exponents for a number of examples. We point out upfront that their exponent  $\kappa_\Gamma$  is exactly the inverse of the value  $\hat{\mu}_\Gamma$  introduced in Def. 1.2 below.

Let  $K$  be any number field whose ring of integers  $\mathcal{O}_K$  is a discrete subring of  $\mathbb{C}$ . In addition, we require that any complex number  $z$  should be within unit distance of some element of  $\mathcal{O}_K$ . The only such rings correspond to the rings of integers for the quadratic number fields  $\mathbb{Q}(\sqrt{-d})$  where  $d = 1, 3, 7$  or  $11$  [see 3, Remark 2.4]. By  $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ , we shall mean the lattice consisting of special linear matrices with entries belonging to  $\mathcal{O}_K$ . Consider its action on the punctured complex plane  $\mathbb{C}^2 \setminus \{0\}$  via matrix multiplication on the left. Abusing notation, we use  $|\cdot|$  with matrices as well as complex numbers which means that some clarification is in order. For any matrix  $A$ , we let  $|A|$  be the maximum of the modulus of its entries while for a complex number,  $|z|$  stands for the Euclidean distance to the origin. We use lowercase Greek and both upper and lowercase Roman letters for various operating matrices and vectors will be in boldface (e. g.  $\mathbf{z}$ ). The following terminology is motivated from Bugeaud and Laurent [1].

**Definition 1.1.** Let  $\mathbf{z}, \mathbf{y} \in \mathbb{C}^2$ . The Diophantine exponent  $\mu_\Gamma(\mathbf{z}, \mathbf{y})$  is the quantity

$$\sup \left\{ \omega \mid |\gamma \mathbf{z} - \mathbf{y}| \leq |\gamma|^{-\omega} \text{ has infinitely many solutions in } \gamma \in \Gamma \right\}.$$

**Definition 1.2.** The exponent  $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})$  stands for the supremum of all  $\omega$ ’s for which the system of inequalities

$$|\gamma \mathbf{z} - \mathbf{y}| \leq T^{-\omega}, \quad |\gamma| \leq T$$

has solutions for *all*  $T$  sufficiently large.

If the results of Dirichlet and Minkowski were to be recast in this language, they will say that the uniform exponents for respectively approximating the points  $(\theta, 0)$  and  $(\theta, \alpha)$  using an integral pair  $(p, q)$  are both  $\geq 1$ . Further, measure - theoretic considerations dictate that both the equalities hold except on some set of Lebesgue measure zero. Prop. 3.12 and the subsequent discussion gives us an analogue of Minkowski’s theorem for approximating a complex pair  $(\xi, z)$  with the help of  $\mathcal{O}_K$  - integers.

It follows from the definitions that for all pairs  $\mathbf{z}, \mathbf{y} \in \mathbb{C}^2$ , we have  $\mu_\Gamma(\mathbf{z}, \mathbf{y}) \geq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})$ . For the analogous situation of  $\mathrm{SL}_2(\mathbb{Z})$  acting on  $\mathbb{R}^2$ , Laurent and Nogueira [8] came up with estimates for the exponents defined above and in some cases, get the lower and upper bounds to be equal, which turn out to be functions of the irrationality measures of the slopes of the starting and the target point. For approximation in our setting, we follow in their footsteps to a large extent. When  $d = 1$  so that  $\mathcal{O}_K$  is the ring of Gaussian integers, a continued fraction expansion algorithm for complex numbers with partial quotients from  $\mathcal{O}_K$  was given by Hurwitz [7]. The latter is made use of for constructing certain *convergent matrices*. The case  $d = 3$  has the ring of Eisenstein integers as its integral ring and we have an analogue in the shape of nearest (Eisenstein) integer algorithm [3]. These help us to approach any fixed target point  $\mathbf{y} \in \mathbb{C}^2$  starting at some “irrational” vector  $\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2$  both of whose coordinates are non-zero and the slope  $\xi = z_1/z_2 \in \mathbb{C}' := \mathbb{C} \setminus K$ . Note that  $\mathbb{C}'$  is a full measure subset of  $\mathbb{C}$  as  $K$  is only countable.

We give the following general definition inspired from that of irrationality measure for real irrational numbers.

**Definition 1.3.** The **K - irrationality measure**  $\omega_K(z)$  for any  $z \in \mathbb{C}'$  is the supremum of all numbers  $\omega$  such that the inequality

$$|qz - p| \leq \frac{1}{|q|^\omega}$$

has infinitely many solutions in  $p \in \mathcal{O}_K$ ,  $q \in \mathcal{O}_K \setminus \{0\}$ .

Sullivan [13, Theorem 1] amongst others has formulated and proved the Khintchine theorem for Diophantine approximation of complex numbers by rationals coming from some fixed imaginary quadratic extension of  $\mathbb{Q}$ . In particular, it implies that for all fields  $K$  being considered here, the irrationality measure  $\omega_K(z)$  is an almost everywhere constant function on  $\mathbb{C}'$  with respect to the induced Lebesgue measure. Using the convergence case of Borel-Cantelli lemma along with Dirichlet's box principle (see [2, pg. 1] and also Lem. 3.2 below), one can independently verify its generic value to be 1 and greater than that everywhere else. At this point, we remind the reader that the exponents  $\mu_\Gamma$  and  $\hat{\mu}_\Gamma$  defined above are invariant under  $\Gamma$  - action and, therefore, constant a. e. owing to the ergodicity of the action.

For non - negative functions  $f$  and  $g$ , the Vinogradov notation  $f \ll g$  (similarly  $f \gg g$ ) means that there exists some  $C > 0$  for which  $f(x) \leq Cg(x)$  for all  $x$  in the domain. The dependence of this implicit constant on some ambient parameters  $a, b, c, \dots$  will be often indicated in the subscript as  $\ll_{a,b,c,\dots}$ . The main result contained in this paper is given below.

**Theorem 1.4.** *Let  $K$  be of the form  $\mathbb{Q}(\sqrt{-d})$  where  $d = 1, 3, 7$  or  $11$ . Also, suppose that a continued fraction algorithm for approximating an arbitrary complex number  $z$  with elements of  $K$  exists and has the following properties for all  $n \gg 0$ ,*

- (1) *the denominators of the convergents rise monotonically, i. e.,  $|q_{n+1}| > |q_n|$ , and*
- (2) *there exists  $r_0 \in \mathbb{N}$  and  $\theta > 1$  for which  $|q_{n+r_0}| \geq \theta|q_n|$ .*

*Then, for the full measure subset  $\{\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2 \mid z_1/z_2 \in \mathbb{C} \setminus K, \omega_K(z_1/z_2) = 1\}$  and  $\Gamma = SL_2(\mathcal{O}_K)$  acting linearly on the complex plane, it is true that:*

- i) the exponent of approximation to the origin is  $\mu_\Gamma(\mathbf{z}, \mathbf{0}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0}) = 1$ ,*
- ii) for almost all target points  $\mathbf{y}$  with the slope  $y = y_1/y_2 \in \mathbb{C} \setminus K$  and  $\omega_K(y) = 1$ ,*

$$(1.5) \quad 1/3 \leq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2, \text{ and}$$

- iii) for target points  $\mathbf{y}$  with slope  $y \in K$ , we have*

$$(1.6) \quad \mu_\Gamma(\mathbf{z}, \mathbf{y}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) = 1/2.$$

While the discreteness of  $\mathcal{O}_K$  (which is ensured by taking  $K = \mathbb{Q}(\sqrt{-d})$ ,  $d$  as above) immediately implies that of  $SL_2(\mathcal{O}_K)$  and is also used in working out the generic  $K$  - irrationality measure  $\omega_K$ , the exponential rise of denominator sizes in the continued fraction algorithm helps in bounding the various intermediate matrices properly.

We emphasize again that the results of Laurent and Nogueira [8] are valid for all starting points  $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$  with irrational slopes and all target points, but we have nice answers only on a full measure subset of starting points where  $\omega_K(z_1/z_2) = 1$ . Furthermore, we get a reasonable lower bound for target points  $\mathbf{y}$  with a  $K$  - irrational slope only when  $\omega_K(y_1/y_2) = 1$  and the upper bound of  $1/2$  in Eq. (1.5) is true for some full measure subset of  $\mathbb{C}^2$  (coming out of the Borel - Cantelli lemma and perhaps depending on  $\mathbf{z}$ ).

A continued fraction theory as assumed in Th. 1.4 is provided for  $d = 1$  and  $3$  in [3, 4, 7]. We discuss and suitably modify some of their statements in the next section. The jugglery

with approximating matrices comes in Sect. 3 which, in various parts, gives us Theorem 1.4 (Prop. 3.1 for (i), Props. 3.8 & 3.10 for (ii) and Prop. 3.14 for (iii), respectively).

## 2. CONTINUED FRACTIONS FOR COMPLEX NUMBERS

Dani and Nogueira [4] have considered a family of continued fraction expansions for complex numbers where the partial quotients  $a_n \in \mathbb{Z}[i]$ , the ring of Gaussian integers. In [3], Dani also dealt with continued fractions in terms of Eisenstein integers  $a + b\zeta$  where  $a, b \in \mathbb{Z}$  and  $\zeta^2 + \zeta = -1$ .

In particular, Dani and Nogueira give the best known results for the rate of approximation by convergents coming from Hurwitz's algorithm. Hurwitz [7] described a simple *nearest integer algorithm* which picks a Gaussian integer  $a$  nearest to any given complex number  $z$  (if there is more than one candidate satisfying the condition, choose any one of them). One then proceeds by induction as

$$(2.1) \quad z_0 = z \text{ and } z_{n+1} = (z_n - a_n)^{-1}.$$

If  $z = [a_0, a_1, \dots]$ , then  $a_i \in \mathbb{Z}[i]$  for all  $i$  and  $|a_i| > 1$  for  $i \geq 1$ . On defining the associated numerator and denominator (of the  $n$ th convergent) sequences of Gaussian integers in a recursive fashion as

$$(2.2) \quad \begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2} \text{ for } n \geq 0, \text{ and} \\ q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2} \text{ for } n \geq 0, \end{aligned}$$

we are assured of the exponential growth of the size of the denominators, namely that  $|q_n| > |q_{n-1}|$  and  $|q_n| \geq \theta |q_{n-2}|$  for all  $n \geq 1$ , where  $\theta = (\sqrt{5} + 1)/2$  [4, Corollary 5.3]. The latter guarantees that the distance between the complex number  $z$  and its  $n$ th convergent is small enough in terms of the size of the denominator  $q_{n+1}$  of the succeeding convergent.

We have a similar situation at hand for the Eisenstein integers. Theorem 4.3 of [3] tells us that for the continued fraction expansion with respect to the nearest integer algorithm, we have the monotonous rise of the denominator sizes as well as  $|q_n| > 4|q_{n-2}|/3$  for  $n \geq 1$ . We now give an alteration of [4, Proposition 2.1].

**Lemma 2.1.** *Let  $R$  be a discrete subring of  $\mathbb{C}$  with  $\text{Frac}(R)$  being its field of fractions. Further, let  $\{a_n\} \subset R$  be a sequence which defines a continued fraction expansion of some  $z \in \mathbb{C} \setminus \text{Frac}(R)$  and  $p_n/q_n$  be the corresponding sequence of convergents for which the hypothesis of Th. 1.4 holds. Then, there exist  $C_1$  and  $n_0$  positive such that*

$$(2.3) \quad |q_n z - p_n| \leq \frac{C_1}{|q_{n+1}|} \quad \forall n > n_0.$$

*Proof.* We need to look more closely at the proof given in [4] which goes through for any discrete ring without any changes whatsoever. There, the authors have argued that

$$(2.4) \quad \left| z - \frac{p_n}{q_n} \right| \leq \sum_{k=0}^{\infty} \frac{1}{|q_{n+k} q_{n+k+1}|} \leq \frac{C_0}{|q_n|^2}, \text{ where } C_0 = \frac{r_0 \theta^2}{\theta^2 - 1}.$$

We separate the first term from the series on the right above

$$(2.5) \quad \frac{1}{|q_n q_{n+1}|} + \sum_{k=1}^{\infty} \frac{1}{|q_{n+k} q_{n+k+1}|} \leq \frac{1}{|q_n q_{n+1}|} + \frac{C_0}{|q_{n+1}|^2}$$

and the upper bound is arrived at by taking  $n = n + 1$  in the last step of their calculation. Multiplying Eq. (2.4) by  $|q_n|$  ( $\neq 0$ ) on both sides and recalling that  $|q_n| < |q_{n+1}|$ , we then

have that the scaled error  $|\epsilon_n| = |q_n z - p_n| \leq C_1 |q_{n+1}|^{-1}$ , where  $C_1 = C_0 + 1$  is an absolute constant.  $\square$

Let us now try to obtain a lower bound for  $\epsilon_n$  which is not a priori available. Unlike the simple continued fractions for real numbers, we get a very weak lower estimate for the  $n$ -th error term. But before that, consider two different convergents  $p_n/q_n$  and  $p_{n+r}/q_{n+r}$  for some  $n \geq 0$ ,  $r > 0$  arising from a continued fraction expansion of a fixed  $z \in \mathbb{C} \setminus K$ , where the associated partial quotients belong to the (discrete) ring of integers  $\mathcal{O}_K$ . Our claim is that the two convergents are not the same complex number. If not, let  $p_{n+r} = \kappa p_n$  and  $q_{n+r} = \kappa q_n$  for some  $\kappa \in \mathbb{C} \setminus \{0\}$ . As  $|q_{n+1}| > |q_n|$  for all  $n$ , we get that  $|\kappa| > 1$ . Also,

$$(2.6) \quad |\kappa(p_n q_{n+r-1} - q_n p_{n+r-1})| = |p_{n+r} q_{n+r-1} - q_{n+r} p_{n+r-1}| = 1$$

and hence, the non-zero complex number  $p_n q_{n+r-1} - q_n p_{n+r-1} \in \mathcal{O}_K$  has absolute value at least 1, the latter being a discrete ring. But, this is a contradiction.

**Lemma 2.2.** *With the same notations and conventions as in Lemma 2.1, there exist  $C_2 > 0$  and  $r_1 \in \mathbb{N}$  such that*

$$|\epsilon_n| \geq \frac{C_2}{|q_{n+r_1}|}.$$

*Proof.* Apply triangle inequality to the three numbers  $z, p_n/q_n$  and  $p_{n+r}/q_{n+r}$  giving

$$(2.7) \quad \left| z - \frac{p_n}{q_n} \right| \geq \left| \frac{p_{n+r}}{q_{n+r}} - \frac{p_n}{q_n} \right| - \left| z - \frac{p_{n+r}}{q_{n+r}} \right| \geq \frac{1}{|q_n| |q_{n+r}|} - \frac{C_0}{|q_{n+r}|^2}$$

when we employ (2.4). This implies that

$$(2.8) \quad |\epsilon_n| = |q_n z - p_n| \geq \frac{1}{|q_{n+r}|} - \frac{|q_n|}{|q_{n+r}|} \frac{C_0}{|q_{n+r}|}.$$

Now,  $|q_{n+r}| \geq \theta |q_{n+r-r_0}| \geq \dots \geq \theta^{\lfloor \frac{r}{r_0} \rfloor} |q_n|$  for all  $n, r$  by our assumption. Thus,

$$(2.9) \quad |\epsilon_n| \geq \frac{1}{|q_{n+r}|} - \frac{C_0}{\theta^{\lfloor \frac{r}{r_0} \rfloor} |q_{n+r}|}$$

As  $\theta > 1$ , the constant in the second term on the right side becomes less than 1 for some  $r_1$  sufficiently large and we get the required lower bound for some constant  $C_2 > 0$  and  $r_1 \in \mathbb{N}$  depending only on  $R$  and the continued fraction algorithm in effect.  $\square$

When the  $K$ -irrationality measure  $\omega_K(z)$  is finite and  $\omega > \omega_K(z)$ , then we must have  $|q_{n+1}| \leq |q_n|^\omega$  for all  $n \geq N_0(\omega)$ . Combining Lemmata 2.1 and 2.2, we conclude that for all  $\omega > \omega_K(z)$ ,

$$(2.10) \quad \frac{C_2}{|q_{n+1}|^{\omega^{r_1-1}}} \leq |\epsilon_n| \leq \frac{C_1}{|q_{n+1}|}$$

for all large enough  $n$ . In addition, we get the usual identity

$$(2.11) \quad q_n p_{n-1} - p_n q_{n-1} = (-1)^n$$

as a bonus from the formal theory of continued fractions. It is this particular property of theirs which enables them to be of good use in constructing the so-called *convergent matrices* discussed in the next section.

### 3. CONVERGENT MATRICES

Let  $\xi \in \mathbb{C} \setminus K$  and let  $p_k/q_k$  for  $p_k, q_k \in \mathcal{O}_K$  denote the convergent of order  $k$  to  $\xi$ , due to some continued fraction expansion algorithm which satisfies the hypothesis of Theorem 1.4. The construction of convergent matrices for the complex setting mimics the one for  $\mathbb{R}^2$ . As in [8], we define the  $k$ -th convergent matrix

$$(3.1) \quad M_k := \begin{pmatrix} q_k & -p_k \\ (-1)^{k-1}q_{k-1} & (-1)^k p_{k-1} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) = \Gamma.$$

The powers of  $-1$  have been inserted so that the matrices are special linear ones once we have (2.11). The supremum norm of the above matrix is  $\max(|q_k|, |p_k|)$  since the size of the denominators increases monotonically, and the numerators  $p_k$ 's should increase accordingly in order to approximate better and better the fixed complex number  $\xi$ . If necessary, we pre-multiply the vector  $\mathbf{z} \in \mathbb{C}^2$  by the  $\mathrm{SL}_2(\mathcal{O}_K)$  matrix

$$(3.2) \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to have the slope  $\xi$  with  $|\xi| \leq 1$  while the size  $|J\mathbf{z}| = |\mathbf{z}|$  remains the same. Also, note that  $|J\gamma| = |\gamma J| = |\gamma|$  for any  $2 \times 2$  matrix  $\gamma$ . Thus, it is alright to take  $|M_k| \asymp |q_k|$  and we will do as much from here on without explicitly saying so. When concerned with the  $\Gamma$ -orbit of the point  $\mathbf{z} = (z_1, z_2)^t$  having  $z_1/z_2 = \xi$ , we see that

$$(3.3) \quad M_k \mathbf{z} = z_2 \begin{pmatrix} \epsilon_k \\ (-1)^{k-1} \epsilon_{k-1} \end{pmatrix}$$

implying that

$$(3.4) \quad |M_k \mathbf{z}| \leq |\mathbf{z}| \frac{C_1}{|q_k|} \ll_K \frac{|\mathbf{z}|}{|M_k|} \text{ as both } |\epsilon_k| \leq \frac{C_1}{|q_{k+1}|} < \frac{C_1}{|q_k|} \text{ and } |\epsilon_{k-1}| \leq \frac{C_1}{|q_k|}$$

leveraging (2.3). Thus, there are infinitely many such matrices and this immediately tells us that  $\mu(\mathbf{z}, \mathbf{0}) \geq 1$ . The proof of  $\mu(\mathbf{z}, \mathbf{0}) \leq 1$  goes along the same lines as [8, Lemma 1] with the necessary modifications by  $|q_k|$  and  $|q_{k+1}|$  replacing  $q_k$  and  $q_{k+1}$ , respectively and having appropriate constants in place. Trivially, we also get an upper bound on  $\hat{\mu}(\mathbf{z}, \mathbf{0})$ . For the reverse inequality, it suffices to consider the matrices  $M_k$  with

$$(3.5) \quad |q_k| \asymp |M_k| \leq T \leq |M_{k+1}| \asymp |q_{k+1}|, \text{ and } |M_k \mathbf{z}| \ll_{\mathbf{z}} \frac{1}{|q_k|} \ll \frac{1}{|q_{k+1}|^{1/\omega}} \ll \frac{1}{T^{1/\omega}}$$

for  $\omega > \omega_K(\xi)$  and all  $k > k_0 = k_0(\xi, \omega)$ . Letting  $\omega_K(\xi) \leftarrow \omega$  from the right, we have

**Proposition 3.1.** *For any vector  $\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2$  with slope  $\xi = z_1/z_2 \in \mathbb{C}'$  such that a continued fraction expansion for  $\xi$  in terms of  $\mathcal{O}_K$ -integers exist and satisfies the conditions of Th. 1.4, we have the exponents of approximation*

$$\mu_\Gamma(\mathbf{z}, \mathbf{0}) = 1 \quad \text{and} \quad \frac{1}{\omega_K(\xi)} \leq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0}) \leq 1.$$

We digress now for a bit to prove a claim we made in the discussion after Def. 1.3.

**Lemma 3.2.** *For  $K = \mathbb{Q}(\sqrt{-d})$  where  $d = 1, 3, 7$  or  $11$ , the  $K$ -irrationality measure  $\omega_K(z)$  is equal to 1 for Lebesgue almost all and  $\geq 1$  for all  $z \in \mathbb{C}'$ .*

*Proof.* Given  $Q > 1$ , the number of  $\mathcal{O}_K$  integers  $q$  with  $|q| \leq Q/2$  is  $\geq c_K Q^2$  for some  $c_K > 0$ . For each of these  $q$ 's, there exists a unique  $p = p(q) \in \mathcal{O}_K$  such that the complex number  $qz - p$  belongs to a fixed fundamental polygon  $\mathcal{F}_K$  for  $\mathcal{O}_K$  in  $\mathbb{C}$ . Therefore, we have at least  $c_K Q^2$  many distinct numbers in  $\mathcal{F}_K$  as  $z \notin K$ . If we now divide this polygon into  $\approx c_K Q^2$  many subpolygons each of which has diameter  $\leq c_K^{-1/2} Q^{-1}$ , by Dirichlet's pigeonhole principle, we should have that one of them contains both  $q_1 z - p_1$  and  $q_2 z - p_2$  for some  $|q_1|, |q_2| \leq Q/2$  and  $q_1 \neq q_2$ . In conclusion,

$$(3.6) \quad |(q_1 - q_2)z - (p_1 - p_2)| \leq \frac{1}{\sqrt{c_K} Q} \ll_K \frac{1}{|q_1 - q_2|}$$

giving us that  $\omega_K(z) \geq 1$ . To see that the equality holds for almost all  $z \in \mathbb{C}'$ , notice the number of  $p \in \mathcal{O}_K$  such that  $p/q \in \mathcal{F}_K$  is  $\leq b_K |q|^2$  for any fixed  $q$  and the number of  $q \in \mathcal{O}_K$  for which  $|q| \sim Q$  is  $\leq b'_K Q$ . Therefore, the series in the Borel-Cantelli lemma for the family of discs of radius  $1/|q|^{1+s}$  around the  $K$ -rational point  $p/q$  is dominated by

$$(3.7) \quad \sum_{Q>1} b_K b'_K \frac{Q^3}{(Q^{1+s})^2}.$$

The latter converges for all  $s > 1$  implying that for  $s$  in this range, the limsup set

$$(3.8) \quad \limsup_{p/q \in K} D\left(\frac{p}{q}, \frac{1}{|q|^{1+s}}\right)$$

has Lebesgue measure zero. In other words,  $\omega_K(z) = 1$  for almost all  $z \in \mathbb{C}'$ .  $\square$

Hence, the generic value (in  $\mathbf{z}$ ) of both  $\mu_\Gamma(\mathbf{z}, \mathbf{0})$  and  $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0})$  is 1. We, thereby, have the first claim of Theorem 1.4.

A function  $h : X \rightarrow [0, \infty)$  on a countable space  $X$  is said to be a *height function* if for each  $Q \geq 0$ , the set  $h^{-1}[0, Q]$  is finite. If  $(\varphi, G)$  is an action of a countable group  $G$  with height function  $h$  on a metric space  $X$ , the exponent  $\mu_\varphi(x, y)$  stands for

$$(3.9) \quad \sup\{\mu \mid \text{dist}(gx, y) < h(g)^{-\mu} \text{ has inf. many solutions in } g\}.$$

The uniform variant  $\hat{\mu}_\varphi(x, y)$  is given in the same fashion for all  $x, y \in X$ . Next, we make a simple and more general observation whose proof is immediate from the definitions.

**Proposition 3.3.** *Let  $(G_1, h_1)$  and  $(G_2, h_2)$  be countable groups with  $h_i$  being a height function on  $G_i$  and  $\rho : G_1 \rightarrow G_2$  be a group homomorphism which respects  $h_1$  in the sense that there exists  $c > 1$  s. t.*

$$(3.10) \quad \frac{1}{c} h_1(g) \leq h_2(\rho(g)) \leq c h_1(g) \quad \forall g \in G_1.$$

*Further, let  $(\varphi_i, G_i)$ ,  $i = 1, 2$  be group actions on a metric space  $X$  and  $\varphi_2 \circ \rho = \varphi_1$ . Then, for all pairs  $x, y \in X$ ,*

$$(3.11) \quad \mu_{\varphi_2}(x, y) \geq \mu_{\varphi_1}(x, y) \text{ and } \hat{\mu}_{\varphi_2}(x, y) \geq \hat{\mu}_{\varphi_1}(x, y).$$

As the group  $SL_2(\mathbb{Z}[i])$  sits inside  $SL_4(\mathbb{Z})$  owing to  $a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  with the height function on  $SL_2(\mathbb{Z}[i])$  preserved and the standard linear actions on  $\mathbb{C}^2 \cong \mathbb{R}^4$  coinciding under the resultant embedding, we have the following corollary for simultaneous approximation by primitive integral vectors in dimension 4 by combining Props. 3.1 and 3.3.

**Corollary 3.4.** *For almost all  $\mathbf{v} \in \mathbb{R}^4$ , the exponents  $\mu(\mathbf{v}, \mathbf{0}_4)$  and  $\hat{\mu}(\mathbf{v}, \mathbf{0}_4)$  for approaching the origin via  $\mathrm{SL}_4(\mathbb{Z})$  orbit, are greater than or equal to 1.*

We will not write down the corresponding true statements for other target points in  $\mathbb{R}^4$ . The next lemma bounds the size of a convergent matrix  $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$  in terms of the entries in its decomposition. The idea here is to bring the starting point  $\mathbf{z}$  sufficiently close to the origin using matrices  $M_k$  as above, spread it around as a lattice with the help of the subgroup

$$(3.12) \quad \mathcal{U} = \left\{ U^\ell := \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \mid \ell \in \mathcal{O}_K \right\},$$

and finally rotate the lattice so obtained by applying matrices  $N_j$  which attempt to take the “complex line”  $\langle (z_1, 0) \rangle$  closer to  $\langle (z_1, \xi z_1) \rangle$ .

**Lemma 3.5** (Laurent and Nogueira [8]). *Let  $k \in \mathbb{N}$  and  $\ell \in \mathcal{O}_K$ . For any arbitrary*

$$(3.13) \quad N = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \in \Gamma,$$

*the matrix  $\gamma = NU^\ell M_k$  satisfies*

$$(3.14) \quad |(\ell q_{k-1} + (-1)^{k-1} q_k) s| - |s' q_{k-1}| \leq |\gamma| \ll |\ell q_{k-1}| |N| + |N| |q_k|.$$

*Proof.* After two matrix multiplications

$$(3.15) \quad \begin{aligned} \gamma &= NU^\ell M_k = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ (-1)^{k-1} q_{k-1} & (-1)^k p_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} tq_k + (-1)^{k-1} q_{k-1}(t\ell + t') & -tp_k + (-1)^k p_{k-1}(t\ell + t') \\ sq_k + (-1)^{k-1} q_{k-1}(s\ell + s') & -sp_k + (-1)^k p_{k-1}(s\ell + s') \end{pmatrix}. \end{aligned}$$

The bottom left entry of the matrix determines the lower bound in the lemma as soon as we employ the triangle inequality. Because we have already reduced to the case  $|\xi| \leq 1$ , for all large enough  $n$  we have  $|p_n| \ll |q_n|$  and then, the upper bound is easy enough.  $\square$

We now take steps towards obtaining bounds for the vector  $\gamma(\xi, 1)^t$ . The lemma below is again due to Laurent and Nogueira. We sketch its proof here to merely point out the minor difference(s) with the real case.

**Lemma 3.6.** *Let  $k, \ell, N$  and  $\gamma = NU^\ell M_k = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}$  be as in Lemma 3.5 and  $y \in \mathbb{C}$ . If  $\delta = sy - t$  and  $\delta' = s'y - t'$ , we have that*

$$|v_1 \xi + u_1 - y(v_2 \xi + u_2)| \ll_K \frac{|\delta \ell + \delta'|}{|q_k|} + \frac{|\delta|}{|q_{k+1}|}.$$

*Proof.* To begin with,

$$(3.16) \quad \begin{aligned} y(v_2 \xi + u_2) - (v_1 \xi + u_1) &= (-1 \ y) \gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} = (-1 \ y) \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} U^\ell M_k \begin{pmatrix} \xi \\ 1 \end{pmatrix} \\ &= (\delta \ \delta') U^\ell \begin{pmatrix} \epsilon_k \\ (-1)^{k-1} \epsilon_{k-1} \end{pmatrix} \\ &= \delta \epsilon_k + (-1)^{k-1} (\delta \ell + \delta') \epsilon_{k-1}, \end{aligned}$$

Since for the continued fraction expansions being studied here, we have  $|\epsilon_n| \ll_K |q_{n+1}|^{-1}$  for all  $n \gg 1$ , the claim follows.  $\square$



If  $(\Lambda_1, \Lambda_2)^t$  is the difference  $\gamma \mathbf{z} - \mathbf{y}$ , then

$$(3.17) \quad \Lambda_1 = z_2(v_1\xi + u_1) - y_1, \quad \Lambda_2 = z_2(v_2\xi + u_2) - y_2.$$

and on further choosing  $y = y_1/y_2$ , we get

$$(3.18) \quad \begin{aligned} |\Lambda_1 - y\Lambda_2| &= |z_2((v_1\xi + u_1) - y(v_2\xi + u_2))| \\ &\ll_K |z_2| \left( \frac{|\delta\ell + \delta'|}{|q_k|} + \frac{|\delta|}{|q_{k+1}|} \right). \end{aligned}$$

Once we bound one of the components (say  $\Lambda_2$ ) and the difference  $|\Lambda_1 - y\Lambda_2|$ , the vector  $(\Lambda_1, \Lambda_2)^t$  is bounded automatically. We proceed to do just that. After a slight adjustment in the proof of Lemma 3.6, we deduce

$$(3.19) \quad \begin{aligned} \Lambda_2 &= z_2(v_2\xi + u_2) - y_2 \\ &= z_2(s\epsilon_k + (-1)^{k-1}(s\ell + s')\epsilon_{k-1}) - y_2 = (-1)^{k-1}z_2s\epsilon_{k-1}(\ell - \rho), \end{aligned}$$

where

$$(3.20) \quad \rho = \frac{(-1)^{k-1}y_2}{z_2s\epsilon_{k-1}} - \frac{(-1)^{k-1}\epsilon_k}{\epsilon_{k-1}} - \frac{s'}{s}$$

helps us to decide the value of  $\ell$  such that  $|\ell - \rho| \leq C_3$  for some constant  $C_3$  depending only on  $\mathcal{O}_K$  (or  $K$  if you will) and  $|\ell| \leq |\rho|$ , having fixed  $M_k$  and  $N$  first.

**3.1. Generic target points.** In the next few pages, we discuss the situation where the target point  $\mathbf{y} = (y_1, y_2)^t \in \mathbb{C}^2$  has slope  $y = y_1/y_2 \in \mathbb{C}' = \mathbb{C} \setminus K$ . As such points constitute a set of full measure in  $\mathbb{C}^2$ , we shall be inferring properties of almost all points in the complex plane. Let  $t_j/s_j$  and  $t_{j-1}/s_{j-1}$  be consecutive convergents in an  $\mathcal{O}_K$ -continued fraction expansion of  $y$  for our fixed target point  $\mathbf{y}$ . As argued for  $\xi$ , we may also suppose  $|y| \leq 1$  thanks to the  $J$  of Eq. (3.2). When

$$(3.21) \quad t = t_j, \quad s = s_j, \quad t' = (-1)^{j-1}t_{j-1} \text{ and } s' = (-1)^{j-1}s_{j-1},$$

the matrix  $N_j$  given by  $\begin{pmatrix} t & t' \\ s & s' \end{pmatrix}$  belongs to  $\Gamma = SL_2(\mathcal{O}_K)$  and for any  $\omega > \omega_K(\xi)$ , the auxiliary term  $\rho$  in Eq. (3.20) is confined within the range

$$(3.22) \quad \frac{1}{C_1} \left| \frac{y_2 q_k}{z_2 s_j} \right| - 2 \leq |\rho| \leq \frac{|y_2| |q_k|^{\omega_{r_1}-1}}{|C_2 z_2 s_j|} + 2$$

for all  $k$  large enough using Lemmata 2.1 and 2.2, the fact that the error term

$$(3.23) \quad \epsilon_n = \frac{(-1)^n}{z_1 \cdots z_{n+1}}$$

with  $|z_i| \geq 1$  for  $i > 0$  [3, Prop. 2.1 (i)] and a monotonous increase in the size of denominators  $s_j$ 's. This in turn tells us that the optimal choice of  $\ell$  obeys

$$(3.24) \quad \frac{1}{C_1} \left| \frac{y_2 q_k}{z_2 s_j} \right| - (C_3 + 2) \leq |\ell| \ll_{\mathbf{y}, \mathbf{z}, K} \frac{|q_k|^{\omega_{r_1}-1}}{|s_j|} + 1.$$

Substituting this in Lemmata 3.5 and 3.6, we have an  $\mathcal{O}_K$ -analogue of [8, Lemma 4].

**Lemma 3.7.** *Let  $j \in \mathbb{N}, k \gg 0$  and  $\omega > \omega_K(\xi)$ . There exists  $\gamma = N_j U^\ell M_k \in \Gamma$  for some  $\ell \in \mathcal{O}_K$  such that*

$$\left| \frac{|y_2|}{|C_1 z_2|} |q_k q_{k-1}| - |s_j q_k| \right| - (C_3 + 3) |s_j q_{k-1}| \leq |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_{k-1}| |q_k|^{\omega_{r_1}-1} + |s_j q_k|$$

as well as

$$(3.25) \quad |\gamma \mathbf{z} - \mathbf{y}| \ll_{\mathbf{y}, \mathbf{z}, K} \frac{|q_k|^{\omega^{r_1-1}-1}}{|s_j s_{j+1}|} + \left| \frac{s_j}{q_k} \right|.$$

*Proof.* The bounds for  $\gamma$  are straightforward. Insofar as  $\gamma \mathbf{z} - \mathbf{y}$  is concerned,

$$(3.26) \quad |\Lambda_2| \leq C_1 C_3 \left| \frac{z_2 s_j}{q_k} \right|,$$

using Eq. (3.19) and an optimal choice of  $\ell$  as explained immediately after Eq. (3.20). Moreover,  $|\Lambda_1| \leq |y \Lambda_2| + |\Lambda_1 - y \Lambda_2| \leq |\Lambda_2| + |\Lambda_1 - y \Lambda_2|$  as  $|y| \leq 1$ . The quantity  $|\Lambda_1 - y \Lambda_2|$  is bounded using Eq. (3.18) while we recall that  $|\delta| \leq C_1/|s_{j+1}|$  and  $|\delta'| \leq C_1/|s_j|$ .  $\square$

With the above lemma on our side, we now make an appropriate choice of the indices  $j$  and  $k$  so that

$$(3.27) \quad |q_{k-1}|^{1/3} < |s_j| \leq |q_k|^{1/3} < |s_{j+1}|.$$

We are assured of the existence of arbitrarily large pairs  $(j, k)$  satisfying the inequalities (3.27) as  $|q_k|$ 's and  $|s_j|$ 's are strictly increasing sequences of real numbers. Such a pair is then fed into the statement of Lemma 3.7 to give us

$$(3.28) \quad |\gamma \mathbf{z} - \mathbf{y}| \ll_{\mathbf{y}, \mathbf{z}, K} \frac{1}{|q_{k-1}|^{1/3} |q_k|^{4/3 - \omega^{r_1-1}}} + \frac{1}{|q_k|^{2/3}}.$$

In this work, we are mostly concerned with the  $\Gamma$ -orbits of generic points in  $\mathbb{C}^2$  whose slope has  $K$ -irrationality measure equal to 1. Thus, it is fair to assume that  $1 \leq \omega_K(\xi) < 3$ , where  $\xi = z_1/z_2 \in \mathbb{C}'$  is the slope of the starting point  $\mathbf{z}$ . For  $\omega > \omega_K(\xi) \geq 1$ , the first term in the sum on the right side of the inequality (3.28) dominates over the second, for all  $k$ 's large enough. Also, for  $\omega > \omega_K(\xi)$ ,  $|q_k| \leq |q_{k-1}|^\omega$  for  $k \gg 0$  where we remind the reader that  $p_k/q_k$ 's are convergents to  $\xi$  coming from a continued fraction expansion algorithm. Furthermore, under the condition (3.27) and for  $\omega < 3$ , the second term in the upper bound for  $|\gamma|$  is much smaller than the first implying that we have the existence of a  $\gamma \in \Gamma$  which satisfies

$$(3.29) \quad |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_k|^{\omega^{r_1-1}+1}, \text{ and } |\gamma \mathbf{z} - \mathbf{y}| \ll_{\mathbf{y}, \mathbf{z}, K} \frac{1}{|q_k|^{\frac{1}{3\omega} + \frac{4}{3} - \omega^{r_1-1}}}.$$

The preceding lemma also tells us that  $|\gamma| \gg |q_k q_{k-1}|$  for the choice of  $j$  and  $k$  according to Eq. (3.27). This ensures the existence of infinitely many matrices  $\gamma \in \text{SL}_2(\mathcal{O}_K)$  satisfying the above system of inequalities.

As hinted before, we could have always started with an  $\omega$  close enough to 1 so that the exponent  $\frac{1}{3\omega} + \frac{4}{3} - \omega^{r_1-1}$  which expresses itself in the upper bound for  $\gamma \mathbf{z} - \mathbf{y}$  in (3.29) is positive. For such an  $\omega > \omega_K(\xi)$ , we therefore have that

$$(3.30) \quad \mu_\Gamma(\mathbf{z}, \mathbf{y}) \geq \frac{1 + 4\omega - 3\omega^{r_1}}{3\omega(\omega^{r_1-1} + 1)}.$$

In the limit  $\omega_K(\xi) \leftarrow \omega$  from the right, and the generic value of the former being 1, we get

**Proposition 3.8.** *For all  $\mathbf{z} \in \mathbb{C}^2 \setminus \{0\}$  having slope  $\xi$  with  $K$ -irrationality measure  $\omega_K(\xi) = 1$  and for all  $\mathbf{y} \in \mathbb{C}^2$  with slope  $y \in \mathbb{C} \setminus K$ ,*

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) \geq 1/3.$$

We now calculate lower bounds for  $\hat{\mu}_\Gamma$ . For almost all target points  $\mathbf{y} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$  with the slope  $y$  belonging to  $\mathbb{C}'$ , we show a mildly stronger result than Prop. 3.8 to be true, i. e.,

$$(3.31) \quad \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \geq 1/3$$

for almost all pairs  $(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$ . In proving this, the  $K$ -irrationality measure  $\omega_K(y)$  associated with  $y$  is used as an auxiliary tool. The result below is the same as [8, Lemma 6] mutatis mutandis and the proof is omitted.

**Lemma 3.9.** *Let  $\omega > \omega_K(\xi) (\geq 1)$  and define*

$$(3.32) \quad \tau := \frac{\omega_K(y)}{2\omega_K(y) + 1} \omega^{r_1-1}.$$

*Given any  $\varepsilon > 0$  and  $k_0 = k_0(\varepsilon) \in \mathbb{N}$ , there exists  $\gamma \in \Gamma$  such that*

$$(3.33) \quad |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_k|^{1+\omega^{r_1-1}} \quad \text{and} \quad |\gamma \mathbf{z} - \mathbf{y}| \leq |q_k|^{\tau-1+\varepsilon}$$

*for all  $k > k_0$ .*

In addition to our assumption that  $\frac{1}{3\omega} + \frac{4}{3} - \omega^{r_1-1}$  is positive, we now also suppose an extra condition that  $\omega < 2^{1/(r_1-1)}$ . This is to ensure that the quantity  $\tau$  defined in Lemma 3.9 remains less than 1. Next, we restrict to  $\varepsilon$  small enough so that for given  $\tau$ , the exponent  $1 - \tau - \varepsilon$  whom we shall meet soon is greater than 0. After this, our recourse is the old but very helpful idea of sandwiching (also used in Laurent and Nogueira [8]) which given any sufficiently large real positive number  $T$ , picks a  $k$  large enough in terms of  $T$  so that

$$(3.34) \quad C|q_k|^{1+\omega^{r_1-1}} \leq T < C|q_{k+1}|^{1+\omega^{r_1-1}},$$

where  $C$  is the hidden constant in the upper bound for  $|\gamma|$  given by Lemma 3.9. Such a choice of  $k$  will mean that both

$$(3.35) \quad |\gamma| \leq T, \quad \text{and} \quad |\gamma \mathbf{z} - \mathbf{y}| \leq \frac{1}{|q_k|^{1-\tau-\varepsilon}} \leq \frac{1}{T^{(1-\tau-\varepsilon)/(\omega+\omega^{r_1})}}$$

hold simultaneously, giving us a lower bound

$$(3.36) \quad \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \geq \frac{1 - \tau - \varepsilon}{\omega + \omega^{r_1}}.$$

As the bound obtained is true for all sufficiently small  $\varepsilon > 0$  and  $\omega > \omega_K(\xi)$ , in the limit  $\varepsilon \rightarrow 0_+$  and  $\omega \rightarrow \omega_K(\xi)_+$ ,

$$(3.37) \quad \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \geq \frac{(2 - (\omega_K(\xi))^{r_1-1}) \omega_K(y) + 1}{(2\omega_K(y) + 1)(\omega_K(\xi) + (\omega_K(\xi))^{r_1})}$$

when the starting point  $\mathbf{z} \in \mathbb{C}^2$  has slope whose  $K$ -irrationality measure  $\omega_K(\xi)$  is very close to that of any generic point in the complex plane. Since we are only concerned with generic pairs  $(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$ , we may as well take both  $\omega_K(\xi)$  and  $\omega_K(y)$  to be equal to 1 whereby  $\hat{\mu}_\Gamma$  comes out to be at least  $1/3$ .

From the literature, we mention results of Pollicott [12] which is about calculating the error term in the equidistribution sum associated with the linear action of cocompact lattices  $\Gamma \subset SL_2(\mathbb{C})$  on  $\mathbb{C}^2$ . The bounds for the generic value of Diophantine exponents then fall out as a corollary. A work in the same spirit for  $SL_2(\mathbb{Z})$  action on  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$  was carried out by Maucourant and Weiss [10] with much weaker estimates than those of Laurent and Nogueira [8]. Applicable in a broader framework, the machinery of Ghosh, Gorodnik, and Nevo [5, 6] is vastly superior and gives the values of exponents for

an array of lattice actions on homogeneous varieties of connected almost simple, semisimple algebraic groups (see, in particular [5, 6]). However, for them too, the lower bound for the uniform exponent  $\hat{\mu}_\Gamma$  as in Def. 1.2 for any given  $\mathbf{z}$  with dense  $\Gamma$ -orbits in the complex plane and a generic target point  $\mathbf{y} \in \mathbb{C}^2$  is off by a factor of 2 compared to ours, as has been told by the authors in a personal communication. Nevertheless, it is one of the papers in this series that we look at next in our search for upper bounds on  $\mu_\Gamma$  and  $\hat{\mu}_\Gamma$ .

Let us turn our attention to Theorem 3.1 of [5]. The punctured plane  $\mathbb{C}^2 \setminus \{\mathbf{0}\}$  can be realized as the special linear group  $\mathrm{SL}_2(\mathbb{C})$  quotiented by the closed upper unipotent subgroup  $H$ . The non-uniform lattice  $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$  acts ergodically on  $G/H$  and we verify that the hypothesis of the theorem is valid in this scenario. In the terminology of Ghosh et al., the coarse volume growth exponent  $a$  for the upper unipotent group  $H \subset \mathrm{SL}_2(\mathbb{C}) = G$  and the lower local dimension  $d'$  of the homogeneous space  $G/H \approx \mathbb{C}^2 \setminus \{\mathbf{0}\}$  equal 2 and 4, respectively. We then have that for any  $\mathbf{z} \in \mathbb{C}^2$  with a dense  $\Gamma$ -orbit and almost all  $\mathbf{y} \in \mathbb{C}^2$ , the inverse

$$(3.38) \quad \kappa(\mathbf{z}, \mathbf{y}) := \frac{1}{\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})} \geq 2$$

which is the same as saying that  $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2$  for all  $\mathbf{z} \in \mathbb{C}^2$  with slope  $\xi \in \mathbb{C}'$  and  $\mathbf{y}$  belonging to a full Lebesgue measure subset (depending on  $\mathbf{z}$ ) of the complex plane. The same proof can be modified to replace  $\hat{\mu}_\Gamma$  with  $\mu_\Gamma$ . To see this, one should apply Borel-Cantelli as soon as we have the estimates for the number of lattice elements  $\gamma \in \Gamma \cap G_t$  of bounded size  $e^t$  and such that  $\gamma\mathbf{z}$  lies within a unit distance of the target point  $\mathbf{y}$ . This is given by  $e^{2t+\varepsilon}$  upto a constant multiple depending on  $\varepsilon$  alone. In summary,

**Proposition 3.10.** *For all pairs  $(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$  with the slopes  $\xi$  of  $\mathbf{z}$  and  $y$  of  $\mathbf{y}$  both having  $K$ -irrationality measure equal to 1, we have*

$$(3.39) \quad 1/3 \leq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}), \text{ and}$$

for all  $\mathbf{z} \in \mathbb{C}^2$  with slope  $\xi \in \mathbb{C}'$  and almost all (depending on  $\mathbf{z}$ ) target point  $\mathbf{y}$ , the upper bound

$$(3.40) \quad \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2.$$

**3.2. Target point with  $K$ -rational slope.** The task of computing exponents is much easier when  $\mathbf{y}$  has slope  $y = y_1/y_2 = a/b \in K$ , where  $a, b \in \mathcal{O}_K$  and  $|\gcd_{\mathcal{O}_K}(a, b)| = 1$ . Without any loss of generality, assume as before that  $\max\{1, |a|\} \leq |b|$ . The column vector  $(a, b)^t$  is taken to be the first column of our matrix  $N$  as the fraction  $a/b$  is the best approximation to  $y$  by  $K$ -rational points. After this step, the second column can be chosen to be some  $(a', b')^t \in \mathcal{O}_K^2$  such that  $ab' - a'b = 1$  and  $|b'| \leq |b|$ . This is possible because of the fact that  $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$  is a Euclidean domain for  $d = 1, 2, 3, 7$  and 11 [11] and then, it is clear that  $|N| \asymp |b|$ . As in Sec. 3.1, let  $\omega > \omega_K(\xi) (\geq 1)$  where  $\xi = z_1/z_2 \in \mathbb{C}'$ . If necessary, we will take  $\omega$  to be very close to  $\omega_K(\xi)$ .

**Lemma 3.11** (cf. [8, Lemma 5]). *Let  $k \in \mathbb{N}$  be large enough. Given  $\mathbf{y} \in \mathbb{C}^2$  with slope  $y \in K$ , there exists some  $\ell \in \mathcal{O}_K$  and  $\gamma = NU^\ell M_k \in \Gamma$  satisfying*

$$(3.41) \quad |q_k q_{k-1}| \ll_{\mathbf{y}, \mathbf{z}, K} |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_{k-1}| |q_k|^{\omega_{r_1-1}} \text{ and also,} \\ |\gamma\mathbf{z} - \mathbf{y}| \ll_K \left| \frac{bz_2}{q_k} \right|.$$

*Proof.* Here, the quantities  $\delta$  and  $\delta'$  defined in Lemma 3.6 equal  $by - a = 0$  and  $b'y - a' = 1/b$ , respectively and thereby,

$$(3.42) \quad |\Lambda_1 - y\Lambda_2| \ll_K \left| \frac{z_2}{bq_k} \right|.$$

For the same reason,  $b$  replaces  $s_j$  in the inequality (3.26) and, thereafter, the triangle inequality gives the required bound on  $|\gamma\mathbf{z} - \mathbf{y}|$ . The same change made to Eq. (3.24) gives us

$$(3.43) \quad |\gamma| \ll |\ell bq_{k-1}| + |bq_k| \ll_{\mathbf{y}, \mathbf{z}, K} |q_{k-1}| |q_k|^{\omega^{r_1-1}} + |q_{k-1}| + |q_k| \ll |q_{k-1}| |q_k|^{\omega^{r_1-1}}$$

in conjunction with Lemma 3.5. On the other hand,

$$(3.44) \quad |\gamma| \gg_{\mathbf{z}} |\ell q_{k-1}| - |q_k| \gg_{\mathbf{y}, \mathbf{z}, K} |q_k q_{k-1}|$$

while again looking at the modified Eq. (3.24), as  $|q_{k-1}| \gg |b|$  for all large enough  $k$ 's.  $\square$

From the bounds on the size of  $\gamma$  in the above lemma, we get

$$(3.45) \quad |q_k| \ll_{\mathbf{y}, \mathbf{z}, K} |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_k|^{\omega^{r_1-1}+1}.$$

The second part in this inequality gives us

$$(3.46) \quad \left| \frac{1}{q_k} \right| \ll_{\mathbf{y}, \mathbf{z}, K} |\gamma|^{-\frac{1}{\omega^{r_1-1}+1}}$$

which in turn implies that

$$(3.47) \quad |\gamma\mathbf{z} - \mathbf{y}| \ll_{\mathbf{y}, \mathbf{z}, K} |\gamma|^{-\frac{1}{\omega^{r_1-1}+1}}$$

for infinitely many  $\gamma \in SL_2(\mathcal{O}_K)$ . This means that the Diophantine exponent  $\mu_\Gamma(\mathbf{z}, \mathbf{y}) \geq (\omega^{r_1-1} + 1)^{-1}$  for all  $\mathbf{y}$  with slope  $y \in K$  and  $\omega > \omega_K(\xi)$ . Taking the limit  $\omega_K(\xi) \leftarrow \omega$  from the right, we conclude that for any starting point  $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$  whose slope  $\xi$  has  $K$ -irrationality measure  $\omega_K(\xi)$  and any target point  $\mathbf{y}$  with “ $K$ -rational slope”,

$$(3.48) \quad \mu_\Gamma(\mathbf{z}, \mathbf{y}) \geq \frac{1}{(\omega_K(\xi))^{r_1-1} + 1}$$

Next on our agenda is a lower bound for  $\hat{\mu}_\Gamma$  when  $\mathbf{y}$  has a  $K$ -rational slope. This value is in general lower than that for  $\mu_\Gamma$  above, but equals the same for almost every  $\mathbf{z}$ . Given  $T \gg 0$  and  $\omega > \omega_K(\xi)$ , we choose  $k$  as

$$(3.49) \quad |q_{k-1}| |q_k|^{\omega^{r_1-1}} \leq T < |q_k| |q_{k+1}|^{\omega^{r_1-1}} \quad \text{whereby} \\ |\gamma| \ll_{\mathbf{y}, \mathbf{z}, K} |q_{k-1}| |q_k|^{\omega^{r_1-1}} \leq T, \text{ and } T \leq |q_k|^{1+\omega^{r_1}}$$

for  $\gamma$  given to us by Lemma 3.11. No more input, apart from repeating the same set of arguments, is required to now deduce that

$$(3.50) \quad \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \geq \frac{1}{(\omega_K(\xi))^{r_1} + 1}.$$

Borrowing an idea from Laurent and Nogueira [9], we get the following transference result

**Proposition 3.12.** *Let  $\xi, y \in \mathbb{C}$ . Then, the exponents for inhomogeneous approximation by  $\mathcal{O}_K$ -integers*

$$\hat{\omega}_K(\xi, y) \geq \frac{1}{(\omega_K(\xi))^{r_1} + 1} \quad \text{and} \quad \omega_K(\xi, y) \geq \frac{1}{(\omega_K(\xi))^{r_1-1} + 1}.$$

*Proof.* In the above observations, let  $\mathbf{z} = (\xi, 1)^t$  and  $\mathbf{y} = (y, y)^t$ . Either of the two rows of the various matrix solutions  $\{\gamma_i\} \subset \mathrm{SL}_2(\mathcal{O}_K)$  thus obtained will do the job.  $\square$

As a special case when  $\omega_K(\xi) = 1$ , we obtain that for all  $\varepsilon > 0$ , there exists a  $T_0 > 0$  and for all  $T > T_0$ , we have a pair  $(q, p) \in \mathcal{O}_K^2$  for which

$$(3.51) \quad |q\xi + p - y| < \frac{1}{T^{1/2-\varepsilon}} \quad \text{and} \quad \max\{|p|, |q|\} \leq T.$$

The lemma written below helps us to obtain an upper bound for the Diophantine exponent  $\mu_\Gamma(\mathbf{z}, \mathbf{y})$  when the starting point has dense  $\mathrm{SL}_2(\mathcal{O}_K)$ -orbit in  $\mathbb{C}^2$  and the target point has a  $K$ -rational slope. The method used is Laurent and Nogueira's factorization technique [8, Theorem 4] to break down any candidate matrix  $\gamma \in \Gamma$  in terms of well-known entities like  $N$  and  $M_k$  in order to be able to say something about the size of  $\gamma\mathbf{z} - \mathbf{y}$  and of the various components appearing in between. As  $\hat{\mu}_\Gamma \leq \mu_\Gamma$ , this will trivially give us an upper bound for  $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})$ .

**Lemma 3.13.** *Let  $\mathbf{z} \in \mathbb{C}^2 \setminus \{0\}$  have a slope  $\xi \in \mathbb{C}'$  and  $\mathbf{y}$  be a fixed target point with slope  $y = a/b \in K$  as introduced in the beginning of the section. For all  $k$  large enough and  $\gamma \in \Gamma$  such that*

$$|\gamma| \leq \frac{1}{3C_1} \left| \frac{y_2}{z_2} \right| |q_k q_{k+1}|, \quad \text{we must have} \quad |\gamma\mathbf{z} - \mathbf{y}| \geq \left| \frac{z_2}{3b} \right| \frac{1}{|q_k|}$$

Here,  $C_1$  refers to the constant discussed in Eq. (2.3), distilled from Dani's continued fraction theory for complex numbers in terms of  $\mathcal{O}_K$ -integers.

*Proof.* Assume, if possible, that for some  $\gamma = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}$  as above, the vector  $\gamma\mathbf{z} - \mathbf{y} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$  has supremum norm strictly less than  $|z_2| |3bq_k|^{-1}$ . Without loss of generality, we may suppose that  $|a| \leq |b|$  because of the matrix  $J$  from Eq. (3.2). Given the complex number  $a/b$  with  $a, b \in \mathcal{O}_K$  and  $|\mathrm{gcd}_{\mathcal{O}_K}(a, b)| = 1$ , we take  $N = \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \in \Gamma$  with  $|b| \leq |N| < 2|b|$ . As in [8], let

$$(3.52) \quad \gamma' := N^{-1}\gamma = \begin{pmatrix} v'_1 & u'_1 \\ v'_2 & u'_2 \end{pmatrix}.$$

Since  $b'y_1 - a'y_2 = y_2/b$ , here too we get that

$$(3.53) \quad \gamma' = \begin{pmatrix} \frac{b'(v_1 y_2 - v_2 y_1)}{y_2} + \frac{v_2}{b} & \frac{b'(u_1 y_2 - u_2 y_1)}{y_2} + \frac{u_2}{b} \\ -\frac{b(v_1 y_2 - v_2 y_1)}{y_2} & -\frac{b(u_1 y_2 - u_2 y_1)}{y_2} \end{pmatrix}$$

after adding and subtracting equal quantities to both the entries in the first row. Also,

$$(3.54) \quad \begin{pmatrix} z_2(v'_1 \xi + u'_1) \\ z_2(v'_2 \xi + u'_2) \end{pmatrix} = \gamma'\mathbf{z} = N^{-1} \left( \mathbf{y} + \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \right) = \begin{pmatrix} \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_2 + a\Lambda_1 \end{pmatrix}.$$

Next, the determinant

$$(3.55) \quad \begin{aligned} v_1 y_2 - v_2 y_1 &= \begin{vmatrix} v_1 & y_1 \\ v_2 & y_2 \end{vmatrix} = \begin{vmatrix} v_1 & \gamma\mathbf{z} - \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \\ v_2 & \end{vmatrix} = z_2 - \begin{vmatrix} v_1 & \Lambda_1 \\ v_2 & \Lambda_2 \end{vmatrix}, \text{ whereby} \\ |v_1 y_2 - v_2 y_1| &\leq |z_2| + 2|\gamma| \max\{|\Lambda_1|, |\Lambda_2|\} \\ &\leq |z_2| + \left| \frac{2y_2}{9C_1 b} q_{k+1} \right| \leq \left| \frac{y_2}{4C_1 b} \right| |q_{k+1}| \end{aligned}$$

for all  $k$  such that  $|q_k| > 36C_1 |bz_2| / |y_2|$ . Combining the last three Eqs. (3.53), (3.54) and (3.55), we conclude that

$$(3.56) \quad \begin{aligned} |v'_2| &= \left| \frac{b}{y_2} (v_1 y_2 - v_2 y_1) \right| < \frac{|q_{k+1}|}{4C_1}, \text{ and} \\ |v'_2 \xi + u'_2| &= \frac{1}{|z_2|} |-b\Lambda_1 + a\Lambda_2| \leq 2 \left| \frac{b}{z_2} \right| \max\{|\Lambda_1|, |\Lambda_2|\} < \frac{2}{3} \cdot |q_k|^{-1} \end{aligned}$$

as  $|a| \leq |b|$ . Now, consider the  $SL_2(\mathcal{O}_K)$  matrix  $g$  defined to be  $N^{-1}\gamma M_k^{-1}$ . Then,  $g = \begin{pmatrix} * & * \\ * & v'_2 p_k + q_k u'_2 \end{pmatrix}$ , and the lower left entry has size

$$(3.57) \quad \begin{aligned} |v'_2 p_k + q_k u'_2| &= |-v'_2(q_k \xi - p_k) + q_k(v'_2 \xi + u'_2)| \\ &\leq \left| \frac{C_1 v'_2}{q_{k+1}} \right| + |q_k| |v'_2 \xi + u'_2| < \frac{1}{4} + \frac{2}{3} < 1. \end{aligned}$$

As the ring of integers  $\mathcal{O}_K$  was taken to be discrete, it has no non-zero element whose Euclidean norm is less than one. This means  $g$  has to be equal to  $\begin{pmatrix} m & \zeta \\ -\zeta^{-1} & 0 \end{pmatrix}$  for some  $m \in \mathbb{Z}$  and  $\zeta \in \mathcal{O}_K^*$  with  $|\zeta| = 1$ . Thereafter, Eq. (3.54) tells us that the vector

$$(3.58) \quad \gamma' \mathbf{z} = \begin{pmatrix} \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_2 + a\Lambda_2 \end{pmatrix} = g M_k \mathbf{z} = z_2 \begin{pmatrix} m\epsilon_k + (-1)^{k-1}\zeta\epsilon_{k-1} \\ -\zeta^{-1}\epsilon_k \end{pmatrix}.$$

We concentrate on the entry in the first coordinate to get

$$(3.59) \quad \begin{aligned} \left| \frac{y_2}{b} \right| - \frac{4}{3} \left| \frac{z_2}{q_k} \right| &\leq \left| \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \right| = |z_2| |m\epsilon_k + (-1)^{k-1}\zeta\epsilon_{k-1}| \\ &\leq C_1 |z_2| \left( \left| \frac{m}{q_{k+1}} \right| + \left| \frac{1}{q_k} \right| \right) \end{aligned}$$

which gives a lower bound on  $|m|$ ,

$$(3.60) \quad |m| \geq \left| \frac{101y_2}{108C_1 bz_2} \right| |q_{k+1}| > 33$$

as  $C_1 > 1$  and we recall that  $|q_{k+1}| > |q_k| > 36C_1 |bz_2| / |y_2|$ . We now have a decomposition for the matrix  $\gamma$  as  $\gamma = NgM_k$  which helps us to get a handle on its size. To be precise,

$$(3.61) \quad \gamma = \pm \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ (-1)^{k-1}q_{k-1} & (-1)^k p_{k-1} \end{pmatrix}$$

so that  $|\gamma| \geq |b|(|m| - 2)|q_k|$  by the triangle inequality. As we have argued that  $|m|$  should be greater than 33 for all suitably large  $k$ 's, the quantity  $|m| - 2$  should be strictly bigger than  $31|m|/33$ . We, therefore, deduce that

$$(3.62) \quad |\gamma| > \frac{31}{33} |bm q_k| \geq \frac{4}{5C_1} \left| \frac{y_2}{z_2} \right| |q_k q_{k+1}|,$$

but the hypothesis was  $|\gamma| \leq \frac{|y_2|}{3C_1 |z_2|} |q_k q_{k+1}|$ , a contradiction.  $\square$

For any  $\gamma \in \Gamma$  with  $|\gamma|$  sufficiently large, pick  $k$  as

$$(3.63) \quad \frac{1}{3C_1} \left| \frac{y_2}{z_2} \right| |q_{k-1} q_k| < |\gamma| \leq \frac{1}{3C_1} \left| \frac{y_2}{z_2} \right| |q_k q_{k+1}|,$$

and for the case when  $\omega_K(\xi)$  is finite, choose any real number  $\omega > \omega_K(\xi)$  so that eventually we have  $|q_{k-1}| \geq |q_k|^{1/\omega}$ . Consequently,

$$(3.64) \quad |\gamma \mathbf{z} - \mathbf{y}| \geq \left| \frac{z_2}{3b} \right| \frac{1}{|q_k|} \gg_{\mathbf{y}, \mathbf{z}} \frac{1}{|\gamma|^{\omega/(\omega+1)}}$$

and letting  $\omega$  approach  $\omega_K(\xi)$  from the right, we get that

$$(3.65) \quad \mu(\mathbf{z}, \mathbf{y}) \leq \frac{\omega_K(\xi)}{\omega_K(\xi) + 1}.$$

The statement is also true for  $\omega_K(\xi) = \infty$ , as can be easily checked. However, as discussed in Section 1, the  $K$ -irrationality measure  $\omega_K(\xi)$  equals 1 for Lebesgue-almost all  $\xi \in \mathbb{C}$ . Therefore, for a full measure subset of  $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ , Eqs. (3.50) and (3.65) together give

**Proposition 3.14.** *For all  $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$  such that the slope  $\xi$  has  $K$ -irrationality measure to be 1, and for all  $\mathbf{y} \in \mathbb{C}^2$  with slope  $y \in K$ ,*

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) = 1/2.$$

This completes the proof of Theorem 1.4.

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